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Statistics of heterodyne detection of Gaussian light

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Abstract. We discuss the statistical properties of the intensity fluctuations of light fields consisting of an incoherent Gaussian component heterodyned with a single-frequency coherent beam. The second moment of the intensity-fluctuation distribution is obtained in an analytic form for the case of a Lorentzian incoherent spectrum for arbitrary values of the bandwidth and frequency displacement from the coherent mode.

The analysis includes as special cases many of the exact results obtained in statistical optics over the last decade.

1. Introduction

In a recent paper (Jakeman and Pike 1968 a, to be referred to as JP) the quantum-mechanical problem of the statistics of homodyne photoelectric detection of Gaussian light was discussed. The intensity-fluctuation distribution and the photon-counting statistics were derived for Lorentzian spectra of arbitrary linewidth. This theory has since been used to determine experimental linewidths of scattered laser light (Jakeman *et al.* 1968). Additional spectral information can be obtained by the use of heterodyne detection where the Gaussian field is superimposed, before detection, on a known coherent component. In this way, for instance, the centre frequency of the model field cited above can also be determined. In many cases the complete spectrum of a Gaussian source can be found by the heterodyne method.

The statistical properties of a Gaussian optical field mixed with a single coherent mode have been studied theoretically by several previous authors. Glauber (1963 b) gave the basic formula for the density operator of superimposed multimode fields. He later found (Glauber 1966), as did also Lachs (1965) independently, the photon-counting distribution and its factorial moments (the actual moments of the intensity-fluctuation distribution) for a superposition of a coherent and a narrow band Gaussian field at the same frequency. Morawitz (1965) calculated $N^{(2)}$ for a superposition of a single coherent mode and a Gaussian component of various line shapes centred at the same frequency. His result can be compared with ours for the case of a Lorentzian profile and it seems that his calculation is restricted to the case of large bandwidth, although this is not explicitly stated. The general heterodyne problem for Gaussian light would include the case where both the centre frequency of the Gaussian component and its spectrum were arbitrary. The theoretical results required for comparison with statistical experiments are the factorial moments $N^{(r)}$ of the photon-counting distribution $p(n, T)$; these can be found from the intensity-fluctuation distribution $P(E)$ of the field or, more conveniently, from its moment-generating function $Q(s)$.

In this paper the spectrum of the Gaussian component is restricted to be of Lorentzian form with half-width at half-height Γ, but this can take arbitrary values compared with the inverse of the sampling time T. The centre frequency ω_0 is also allowed to differ arbitrarily from the coherent signal frequency ω_q. The detector, however, is assumed to respond uniformly over the frequencies involved. The analysis to be presented includes as special cases many of the exact results obtained in statistical optics over the last decade. A guide to the literature is given in table 1, in pictorial form, with names and dates of authors against the particular field or field combinations considered. We have attempted to select the authors first responsible for each new result; although our search of the literature has been fairly thorough, there may be errors in the table of which we should be very grateful to hear. The table does not include the numerous results obtained by superimposing combinations of the same or different distributions after detection. Such combinations might be obtained, for instance, by using different areas of the same photocathode for each component; the simplest example is the superposition of orthogonal components of unpolarized light. Results of this nature can be simply obtained by multiplying together the separate generating functions where they are available. The cases where coherent signals are present at more than one frequency are also excluded.

In § 2 the moment-generating function is calculated and from this in § 4 the second factorial moment of the photon-counting distribution is derived. Less general results are discussed in § 3 where all the limiting cases of the table are obtained. The paper is concluded with a discussion of results in § 5. Much of the detailed mathematics is reserved for the appendix.

2. The moment-generating function

The positive-frequency part of the electric field obtained by the superposition of incoherent Gaussian light and a coherent beam of frequency ω_q may be expanded in normal modes as follows (Glauber 1963 b):

$$\mathscr{E}^+(r,t) = \sum_k \alpha_k e_k(r,t) + \beta_q e_q(r,t) \tag{1}$$

where

$$e_k(r,t) = \mathrm{i}(\tfrac{1}{2}\hbar\omega_k)^{1/2} u_k(r)\exp(-\mathrm{i}\omega_k t). \tag{2}$$

Here the α_k are statistically independent random variables:

$$\langle \alpha_i^* \alpha_j \rangle = \langle n_i \rangle \delta_{ij} \tag{3}$$

with the probability distribution

$$p(\alpha_k) = \frac{1}{\pi\langle n_k\rangle}\exp\left(-\frac{|\alpha_k|^2}{\langle n_k\rangle}\right) \tag{4}$$

and $|\beta_q|^2$ is a fixed intensity associated with the coherent part of the field. Defining the incoherent and coherent parts of the field respectively by

$$\mathscr{E}_\mathrm{i}^+(r,t) = \sum_k \alpha_k e_k(r,t), \qquad \mathscr{E}_\mathrm{c}^+(r,t) = \beta_q e_q(r,t) \tag{5}$$

we may show, using (3) above, that

$$\langle \mathscr{E}^+(r,t)\mathscr{E}^-(r,t)\rangle = \langle \mathscr{E}_\mathrm{i}^+(r,t)\mathscr{E}_\mathrm{i}^-(r,t)\rangle + \langle \mathscr{E}_\mathrm{c}^+(r,t)\mathscr{E}_\mathrm{c}^-(r,t)\rangle. \tag{6}$$

The integrated intensity $E(T)$ is defined by

$$E(T) = \int_0^T \mathscr{E}^+(t)\mathscr{E}^-(t)\,\mathrm{d}t \tag{7}$$

so that we may write

$$\langle E(T)\rangle = \langle E_\mathrm{i}(T)\rangle + \langle E_\mathrm{c}(T)\rangle. \tag{8}$$

We have dropped the r dependence as we consider henceforth the field at a single space point only. It is convenient, following the treatment of JP, to introduce the new basis set $\phi_i(t)$:

$$e_k(t) = \sum_i S_{ki}\phi_i(t) \tag{9}$$

which are orthogonal over the interval $0 \leqslant t \leqslant T$ and chosen so that the a_k's defined by

$$a_k = \sum_i \alpha_i S_{ik} \tag{10}$$

are statistically independent:

$$\langle a_i a_j \rangle = \langle m_i \rangle \delta_{ij} \tag{11}$$

with the probability distribution

$$p(a_k) = \frac{1}{\pi\langle m_k\rangle} \exp\left(-\frac{|a_k|^2}{\langle m_k\rangle}\right). \tag{12}$$

The integrated intensity now takes the form

$$E(T) = \sum_k |a_k + \beta_q S_{qk}|^2 \tag{13}$$

and the moment-generating function $Q(s)$ associated with the intensity-fluctuation distribution $P(E)$ is given by the formula, obtained by integration as in JP,

$$Q(s) = \langle e^{-Es}\rangle = \prod_k \frac{\exp\{-s|\beta_q S_{qk}|^2/(1+s\langle m_k\rangle)\}}{1+s\langle m_k\rangle} \tag{14}$$

where the $\langle m_k\rangle$ are eigenvalues of the integral equation

$$\int_0^T \langle \mathscr{E}_1^{+}(t)\mathscr{E}_1^{-}(t')\rangle\phi_k(t')\,\mathrm{d}t' = \langle m_k\rangle\phi_k(t) \tag{15}$$

and

$$S_{qk} = \int_0^T e_q(t)\phi_k^{*}(t)\,\mathrm{d}t. \tag{16}$$

If the spectrum of the incoherent part of the field is a Lorentzian centred on the frequency ω_0 and of half-width at half-height Γ, the generating function is determined by the relations

$$\int_0^T \exp(-\Gamma|t-t'|)\Phi_k(t')\,\mathrm{d}t' = \frac{\langle m_k\rangle T}{\langle E_i\rangle}\Phi_k(t) \tag{17}$$

$$|\beta_q S_{qk}|^2 = \frac{\langle E_c\rangle}{T}\left[\int_0^T \exp\{\mathrm{i}(\omega_q-\omega_0)t\}\Phi_k^{*}(t)\,\mathrm{d}t\right]^2. \tag{18}$$

The solution of the integral equation has been given by several authors (see, for example, Davenport and Root 1958, p. 99) and the eigenvalues take the form

$$\langle m_k\rangle = \frac{2\langle E\rangle\gamma}{y_k^2+\gamma^2} \tag{19}$$

where $\gamma = \Gamma T$ and the y_k may be divided into two classes satisfying the transcendental equations

$$y_k \tan\tfrac{1}{2}y_k = \gamma, \qquad \hat{y}_k \cot\tfrac{1}{2}\hat{y}_k = -\gamma. \tag{20}$$

The corresponding eigenfunctions are

$$\Phi_k(t) = \left(\frac{2}{T}\right)^{1/2}\frac{\cos(y_k t/T)}{(1+\sin y_k/y_k)^{1/2}}, \qquad \hat{\Phi}_k(t) = \left(\frac{2}{T}\right)^{1/2}\frac{\sin(\hat{y}_k t/T)}{(1-\sin\hat{y}_k/\hat{y}_k)^{1/2}} \tag{21}$$

and it is not difficult to show that this leads to

$$|\beta_q S_{qk}|^2 = \frac{8\langle E_c\rangle y_k^2\gamma^2}{Q_1^2(\Omega)(\Omega^2-y_k^2)(y_k^2+\gamma^2+2\gamma)} \tag{22}$$

$$|\beta_q \hat{S}_{qk}|^2 = \frac{8\langle E_c\rangle^2\hat{y}_k^2\Omega^2}{\hat{Q}_1^2(\Omega)(\Omega^2-\hat{y}_k^2)(\hat{y}_k^2+\gamma^2+2\gamma)} \tag{23}$$

where

$$Q_1^{-1}(\Omega) = \cos\tfrac{1}{2}\Omega - \frac{\Omega}{\gamma}\sin\tfrac{1}{2}\Omega \tag{24}$$

$$\hat{Q}_1^{-1}(\Omega) = \cos\tfrac{1}{2}\Omega + \frac{\gamma}{\Omega}\sin\tfrac{1}{2}\Omega \tag{25}$$

and

$$\Omega = (\omega_q - \omega_0)T. \tag{26}$$

Using the identity (Slepian 1958)

$$\prod_k \frac{1}{1+s\langle m_k\rangle} = Q_1(y)\hat{Q}_1(y)\,\mathrm{e}^{\gamma} \tag{27}$$

where

$$y = (-\gamma^2 - 2\langle E\rangle\gamma s)^{1/2} \tag{28}$$

it is shown in the appendix that the moment-generating function may be expressed in the closed form

$$Q(s) = \mathrm{e}^{\gamma}Q_1(y)\hat{Q}_1(y)Q_2(y)\hat{Q}_2(y). \tag{29}$$

Q_2 and $\hat{Q}_2$ are functions of γ, $s\langle E_i\rangle$, $s\langle E_c\rangle$ and Ω defined in the appendix but not reproduced here owing to their complexity.

3. Some limiting cases

The generating function defined by (29) above is the Laplace transform of the exact intensity-fluctuation distribution function for superposed Gaussian–Lorentzian and single-frequency coherent light. The formula holds for arbitrary values of the three parameters ω_0, Γ and ω_q entering into the theory and can in principle be used to obtain the photon-counting probability distribution and its moments for any values of these quantities. However, it is useful at this stage to consider various limiting situations which lead to algebraic simplification and which may often be realized in practice. Contact with previous work can also be achieved in this way and acts as a useful check on the considerable amount of algebra outlined in the appendix.

In the first column of table 1 the various limiting cases which can be derived from the problem treated here are pictured, followed by mathematical statements of the limits and a bibliography classified according to the quantities actually calculated. The most general case, for which (29) is the generating function, is shown in the first line and we shall in fact use the latter to evaluate the second factorial moment of the intensity-fluctuation distribution in the next section. The somewhat less general case, $\omega_q = \omega_0$, which has not been treated before is characterized by the generating function

$$Q(s) = \mathrm{e}^{\gamma}Q_1(y)\hat{Q}_1(y)Q_3(s) \tag{30}$$

where

$$Q_3(s) = \exp\left[\frac{4s^2\langle E_i\rangle\,\mathrm{d}\{\ln Q_1(y)\}/\mathrm{d}s}{(1-s\langle E_i\rangle)(\gamma+2s\langle E_i\rangle)}\right]\exp\left\{\frac{2s\langle E_c\rangle}{(2+\gamma)(1-s\langle E_i\rangle)}\right\}$$
$$\times\exp\left\{-\frac{s\langle E_c\rangle\gamma(\gamma+4)}{(2+\gamma)(\gamma+2s\langle E_i\rangle)}\right\} \tag{31}$$

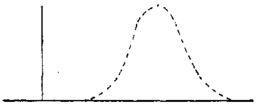
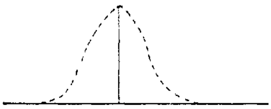
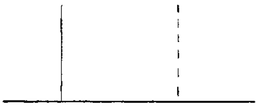
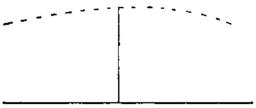
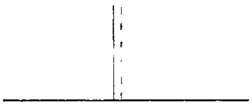
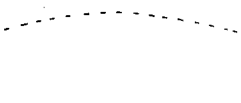
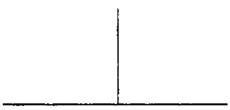
and Q_1, $\hat{Q}_1$, y are defined in the last section. The corresponding second moment is given in the next section. A further case which has not been studied before is shown in the third line of the table. This is the narrow band limit. When $\gamma \to 0$ the generating function (29) reduces to

$$Q(s) = \exp\left\{\frac{(4\langle E_i\rangle\langle E_c\rangle s^2\sin^2\frac{1}{2}\Omega)}{\Omega^2(1+s\langle E_i\rangle)} - \langle E_c\rangle s\right\}\frac{1}{1+s\langle E_i\rangle} \tag{32}$$

and leads to the intensity-fluctuation distribution

$$P(E) = \frac{1}{\langle E_i\rangle}\exp\left\{-\left(E+\frac{8\langle E_c\rangle\sin^2\frac{1}{2}\Omega}{\Omega^2}-\langle E_c\rangle\right)\frac{1}{\langle E_i\rangle}\right\}$$
$$\times I_0\left[\frac{4}{\langle E_i\rangle}\frac{\sin\frac{1}{2}\Omega}{\Omega}\left\{\langle E_c\rangle\left(E+\frac{4\langle E_c\rangle\sin^2\frac{1}{2}\Omega}{\Omega^2}-\langle E_c\rangle\right)\right\}^{1/2}\right]. \tag{33}$$

Table 1. Photon counting statistics of Gaussian–Lorentzian and coherent fields—a bibliography

Field	Limit	$Q(s)$	$P(E)$	$p(n, T)$	$N^{(2)}$	$N^{(r)}$
		present paper			present paper	
	$\omega_q = \omega_0$	present paper			present paper	
	$\Gamma \to 0$	present paper	present paper		present paper	
	$\Gamma \to \infty$ $\omega_q = \omega_0$	present paper	present paper	present paper	present paper	present paper
	$\Gamma \to 0$ $\omega_q = \omega_0$	G66	R45† P67	G66 L65	L65	R45† G66 L65¶
	$\langle E_c \rangle = 0$	H64	JP68 a‖	B66¶ JP68 a‖	B66‖ JP68 a	B66‖
	$\langle E_o \rangle = 0$ $\Gamma \to 0$	G65	R45† M64	M59	R45† M58	R45† G66 L65¶
	$\langle E_o \rangle = 0$ $\Gamma \to \infty$	G65‡ H64‡	JP68 a	M59‡ MP65 JP68 b††	Pu56§ M58§ G65	G65
	$\langle E_l \rangle = 0$	G65	R45† M64	G63 a M64	R45† G65	R45† G65

- - - - Incoherent field.
—— Coherent field.
† Mathematically equivalent calculation.
‡ Not formal expansions in $1/\gamma$.
§ In these early papers bunching was thought to be a property of photons.
¶ Recurrence relations only.
‖ Calculated numerically.
†† To second order in $1/\gamma$.
B, Bédard; G, Glauber 1963 a; H, Helstrom; JP, Jakeman and Pike; L, Lachs; M, Mandel; MP, McLean and Pike; P, Perina; Pu, Purcell; R, Rice.

The second factorial moment for this case is given by

$$N^{(2)} = \frac{\alpha^2\, \mathrm{d}^2 Q(s)}{\mathrm{d}s^2}\bigg|_{s=0}$$

$$= 2\langle n_\mathrm{i}\rangle^2 + 2\langle n_\mathrm{i}\rangle\langle n_\mathrm{c}\rangle\left(1+\frac{4\sin^2\frac{1}{2}\Omega}{\Omega^2}\right) + \langle n_\mathrm{c}\rangle^2 \tag{34}$$

where α is the quantum efficiency of the detector and $\langle n\rangle = \alpha\langle E\rangle$.

Contact with previous work on superposed coherent and incoherent fields is made when $\omega_q = \omega_0$ in the broad and narrow-band limits shown in the fourth and fifth lines of the table respectively. The broad-band limit $\gamma \to \infty$ can be formally obtained, when $\omega_q = \omega_0$, by expanding (30) above in powers of $1/\gamma$. Retaining terms up to first order only gives

$$Q(s) \sim \exp\{-s(\langle E_\mathrm{i}\rangle + \langle E_\mathrm{c}\rangle)\}\left\{1+\frac{s^2\langle E_\mathrm{i}\rangle}{2\gamma}(\langle E_\mathrm{i}\rangle + 4\langle E_\mathrm{c}\rangle)\right\} \tag{35}$$

and leads to the following formulae for the photon-counting distributions and factorial moments:

$$p(n, T) = (-\alpha)^r\, \mathrm{d}^r Q(s)/\mathrm{d}s^r|_{s=1}$$

$$= \frac{\langle n\rangle^2}{n!}\exp(-\langle n\rangle)\left[1+\frac{\langle n_\mathrm{i}\rangle}{2\gamma\langle n\rangle^2}(\langle n_\mathrm{i}\rangle + 4\langle n_\mathrm{c}\rangle)\{(n-\langle n\rangle)^2 - n\}\right] \tag{36}$$

$$N^{(r)} = (-\alpha)^r\, \mathrm{d}^r Q(s)/\mathrm{d}s^r|_{s=0}$$

$$= \frac{\langle n\rangle^r\{1+\langle n_\mathrm{i}\rangle(\langle n_\mathrm{i}\rangle + 4\langle n_\mathrm{c}\rangle)r(r-1)\}}{2\gamma\langle n\rangle^2} \tag{37}$$

where $\langle n\rangle = \langle n_\mathrm{i}\rangle + \langle n_\mathrm{c}\rangle$. As mentioned in the introduction, Morawitz (1965) has given an expression similar to (37) for $N^{(2)}$. There is in fact a slight numerical difference which we take to be an error in Morawitz's work. In the narrow-band limit $\gamma \to 0$ the generating function reduces to the particularly simple form

$$Q(s) = \exp\left(-\frac{s\langle E_\mathrm{c}\rangle}{1+s\langle E_\mathrm{i}\rangle}\right)\frac{1}{1+s\langle E_\mathrm{i}\rangle} \tag{38}$$

in agreement with Glauber (1966) and Perina (1968). Inverse Laplace transformation leads to

$$P(E) = \frac{1}{\langle E_\mathrm{i}\rangle}\exp\left\{-\frac{(E+\langle E_\mathrm{c}\rangle)}{\langle E_\mathrm{i}\rangle}\right\} I_0\left\{\frac{2(E\langle E_\mathrm{c}\rangle)^{1/2}}{\langle E_\mathrm{i}\rangle}\right\} \tag{39}$$

and for the photon-counting distributions and factorial moments we obtain

$$p(n) = \frac{\langle n_\mathrm{i}\rangle^n}{(1+\langle n_\mathrm{i}\rangle)^{n+1}}\exp\left(-\frac{\langle n_\mathrm{c}\rangle}{1+\langle n_\mathrm{i}\rangle}\right)\mathrm{L}_n\left\{-\frac{\langle n_\mathrm{c}\rangle}{\langle n_\mathrm{i}\rangle(1+\langle n_\mathrm{i}\rangle)}\right\} \tag{40}$$

$$N^{(r)} = r!\langle n_\mathrm{i}\rangle^r \mathrm{L}_r\left(-\frac{\langle n_\mathrm{c}\rangle}{\langle n_\mathrm{i}\rangle}\right). \tag{41}$$

These results, expressed in terms of Laguerre polynomials L_n, are identical with those of the earlier work of Glauber (1966), Lachs (1965), Perina (1967) and Rice (1945).

Finally, the single-field cases appearing on the last four lines of the table follow immediately on setting $\langle E_\mathrm{c}\rangle$ or $\langle E_\mathrm{i}\rangle$ to zero in (29) and are seen to be in agreement with previous work.

4. The second factorial moment

Although the intensity-fluctuation distribution cannot be obtained analytically from the formula (29) for $Q(s)$, $p(n, T)$ and $N^{(r)}$ can, in principle, be generated by straightforward differentiation. However, the complex nature of the expression renders this process impracticable beyond second order. In this section, therefore, we restrict ourselves to a derivation of the second factorial moment of the counting distribution.

Using the definition of $N^{(2)}$ appearing in equation (34) and from (14) for $Q(s)$, it is not difficult to show that

$$N^{(2)} = \langle n_i \rangle^2 \left\{1+\frac{1}{\gamma}-\frac{1}{2\gamma^2}+\frac{e^{-2\gamma}}{2\gamma^2}\right\}+2\langle n_i\rangle\langle n_c\rangle+\langle n_c\rangle^2+2\alpha^2\sum_k \langle m_k\rangle |\beta_q S_{qk}|^2. \quad (42)$$

In deriving (42) we have used the property that (14) is equal to the product of the generating function in the absence of $\langle E_c \rangle$ (whose derivatives at $s = 0$ are known) and a factor involving $|\beta_q S_{qk}|$. The last term is defined by (19) and (22) and may be evaluated by writing it as the sum of two terms, each expressible in the form

$$S = \frac{d}{d\Omega^2}\sum_k \frac{y_k^2}{(y_k^2+\gamma^2+2\gamma)(y_k^2+\gamma^2)(y_k^2-\Omega^2)}. \quad (43)$$

The summand can be split into partial fractions

$$S = \frac{d}{d\Omega^2}\sum_k \left\{-\frac{(2+\gamma)}{4(\Omega^2+\gamma^2+2\gamma)}\frac{1}{y_k^2+\gamma^2+2\gamma}+\frac{\gamma}{4(\Omega^2+\gamma^2)}\frac{1}{y_k^2+\gamma^2} + \frac{\Omega^2}{2(\Omega^2+\gamma^2)(\Omega^2+\gamma^2+2\gamma)}\frac{1}{y_k^2-\Omega^2}\right\} \quad (44)$$

and summed following the procedures adopted in the appendix. For example, the first term of S may be written as

$$-\frac{(2+\gamma)}{8\gamma(\Omega^2+\gamma^2+2\gamma)}\frac{d}{d\gamma}\sum_k \ln\left(\frac{1}{1-\langle m_k\rangle\gamma/2\langle E_i\rangle}\right) = -\frac{(2+\gamma)}{8\gamma(\Omega^2+\gamma^2+2\gamma)}\frac{d}{d\gamma}\ln \hat{Q}_1\ (y=0)$$
$$= -\frac{(2+\gamma)}{16\gamma(\Omega^2+\gamma^2+2\gamma)}.$$

The final result is

$$N^{(2)} = \langle n_i\rangle^2\left(1+\frac{1}{\gamma}-\frac{1}{2\gamma^2}+\frac{e^{-2\gamma}}{2\gamma^2}\right)+2\langle n_i\rangle\langle n_c\rangle\left[1+\frac{2e^{-\gamma}\{(\gamma^2-\Omega^2)\cos\Omega-2\gamma\Omega\sin\Omega\}}{(\Omega^2+\gamma^2)^2} + \frac{2\gamma}{\Omega^2+\gamma^2}+\frac{2(\Omega^2-\gamma^2)}{(\Omega^2+\gamma^2)^2}\right]+\langle n_c\rangle^2. \quad (45)$$

When the coherent mode is superimposed on an incoherent field of the same peak frequency, $\omega_0 = \omega_q$ $(\Omega \to 0)$, this expression simplifies:

$$N^{(2)} = \langle n_i\rangle^2\left(1+\frac{1}{\gamma}-\frac{1}{2\gamma^2}+\frac{e^{-2\gamma}}{2\gamma^2}\right)+2\langle n_i\rangle\langle n_c\rangle\left(1+\frac{2e^{-\gamma}}{\gamma^2}+\frac{2}{\gamma}-\frac{2}{\gamma^2}\right)+\langle n_c\rangle^2. \quad (46)$$

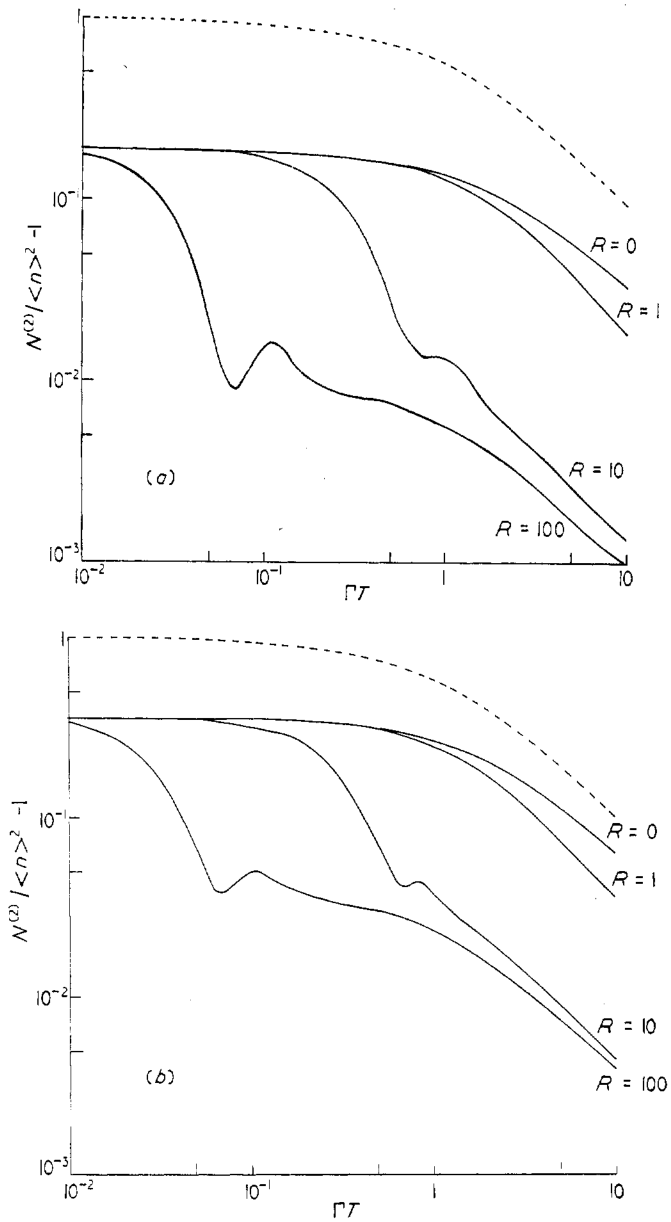
Equation (45) also simplifies if the coherent mode lies well out in the tail of the Lorentzian spectrum of the incoherent field, so that the condition $\gamma \ll \Omega$ is satisfied. In this case $N^{(2)}$ takes the form

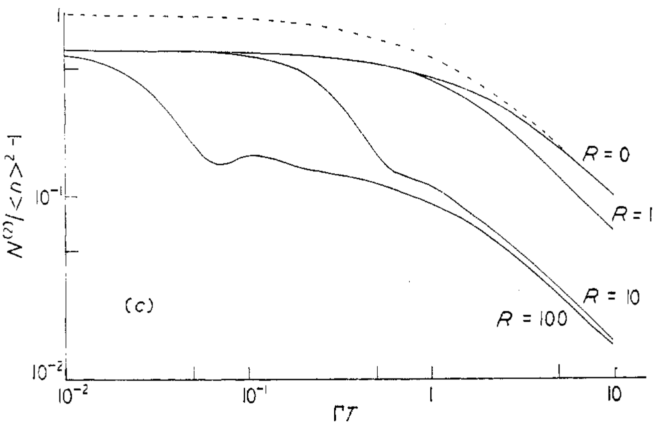
$$N^{(2)} = \langle n_i\rangle^2\left(1+\frac{1}{\gamma}-\frac{1}{2\gamma^2}+\frac{e^{-2\gamma}}{2\gamma^2}\right)+2\langle n_i\rangle\langle n_c\rangle\left\{1+\frac{2}{\Omega^2}(1-e^{-\gamma}\cos\Omega)\right\}+\langle n_c\rangle^2 \quad (47)$$

which in the narrow band limit, $\gamma \to 0$, reduces to (34).

5. Discussion of results

In figure 1 the excess normalized second factorial moment, $N^{(2)}/\langle n\rangle^2-1$, is plotted against γ for various values of the fraction of incoherent light $\langle n_1\rangle/\langle n\rangle$ and R, the ratio of the difference frequency $|\omega_0-\omega_q|$ to the incoherent bandwidth Γ. In all cases the gradients of the curves approach -1 as γ becomes large, whilst for small γ formula (34) is obeyed. The striking new feature to appear in the heterodyne case is the interference pattern evident when the incoherent light does not overlap the coherent mode, i.e. for large R. The origin of the fluctuations in $N^{(2)}$ can be understood from the following considerations. Although the average value of the quantity E defined by equation (7) shows no interference effects in the usual sense because the amplitudes of the incoherent modes are random, fluctuations in the second moment $\langle E^2\rangle$ do not average out. For a single incoherent mode at $\omega=\omega_0$, which is equivalent to the narrow-band limit given by





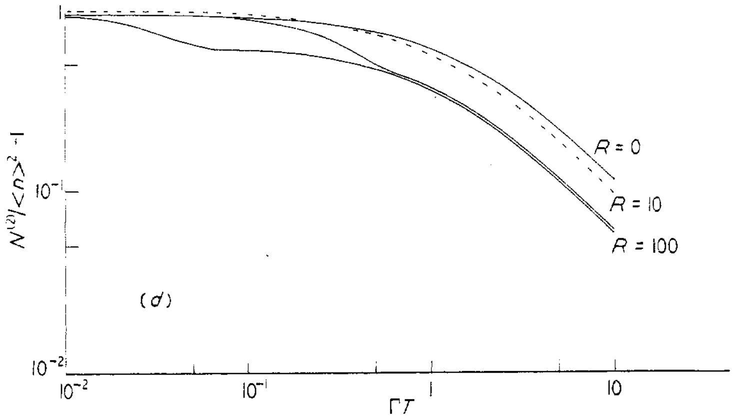


Figure 1. The excess normalized second factorial moment for the values of R ($= |\omega_0 - \omega_q|/\Gamma$) shown for (*a*) $\langle n_i\rangle/\langle n\rangle = 0{\cdot}1$, (*b*) $\langle n_i\rangle/\langle n\rangle = 0{\cdot}2$, (*c*) $\langle n_i\rangle/\langle n\rangle = 0{\cdot}4$, (*d*) $\langle n_i\rangle/\langle n\rangle = 0{\cdot}8$. The broken line gives the value for the case where the coherent signal is absent.

equation (34), the interference term takes the form

$$\frac{8\langle n_i\rangle\langle n_c\rangle \sin^2\{\frac{1}{2}(\omega_0 - \omega_q)T\}}{|\omega_0 - \omega_q|^2 T^2}.$$

This expression has a minimum at $\frac{1}{2}(\omega_0 - \omega_q)T = \pi$ and a maximum near $\frac{1}{2}(\omega_0 - \omega_q)T = 3\pi/2$. For fixed values of R this implies a minimum in $N^{(2)}/\langle n\rangle^2 - 1$ at $\gamma = 2\pi/R$ and a maximum near $3\pi/R$. Inspection of figure 1 confirms these predictions. The interference at higher values of γ is masked by the broadening of the incoherent spectrum (or the equivalent protraction of the time averages of $E(T)$).

In a typical experiment Γ, ω_0 and ω_q would be fixed and ω_q known. R would thus be fixed and experimental measurements of $N^{(2)}/\langle n\rangle^2 - 1$ for two or more values of T would enable the appropriate curve of figure 1 to be chosen and the value of γ and hence Γ determined. Finally, since ω_q is known, the additional parameter ω_0, the centre frequency of the noise component, could be found. Although we have restricted our detailed considerations here to the case of Gaussian–Lorentzian light, a complete curve of $N^{(2)}$ against T in a heterodyne experiment can be used, in principle, to determine an arbitrary spectrum. We shall discuss this problem in a future publication.

Appendix

We shall express the factor $Q_2(y)$ appearing in equation (29) in terms of elementary functions. $\hat{Q}_2(y)$ may be evaluated in a similar way.

From (14), (19) and (20)

$$Q_2(y) = \prod_k \exp\left(-\frac{s|\beta_q S_{qk}|^2}{1+s\langle m_k\rangle}\right) \tag{A1}$$

where

$$\langle m_k\rangle = \frac{2\langle E_i\rangle\gamma}{y_k^2+\gamma^2}, \qquad y_k \tan\tfrac{1}{2}y_k = \gamma. \tag{A2}$$

Taking logarithms of both sides of (A1), we obtain

$$\ln Q_2(y) = \frac{4\langle E_c\rangle}{Q_1^2(\Omega)}\frac{\mathrm{d}}{\mathrm{d}\Omega^2}(\Omega^2 S) \tag{A3}$$

where

$$S = \sum_k \left\{\left(1+\frac{\langle E_i\rangle}{\langle m_k\rangle}\right)\left(\Omega^2+\gamma^2-\frac{2\langle E_i\rangle\gamma}{\langle m_k\rangle}\right)(1+s\langle m_k\rangle)\right\}^{-1}. \tag{A4}$$

This sum can be split up into partial fractions and we must evaluate sums of the three types

$$S_1 = \sum_k \frac{s\langle m_k\rangle}{1+s\langle m_k\rangle}, \qquad S_2 = \sum_k \frac{1}{1+\langle E_i\rangle/\langle m_k\rangle}, \qquad S_3 = \sum \frac{1}{\Omega^2+\gamma^2-2\langle E_i\rangle\gamma/\langle m_k\rangle}.$$

Now

$$S_1 = s\frac{\mathrm{d}}{\mathrm{d}s}\sum_k \ln(1+s\langle m_k\rangle)$$

$$= s\frac{\mathrm{d}}{\mathrm{d}s}\ln\prod_k(1+s\langle m_k\rangle)$$

which may be evaluated immediately, using relation (27), to give

$$S_1 = -s\frac{\mathrm{d}}{\mathrm{d}s}\ln Q_1(y). \tag{A5}$$

Similarly

$$S_3 = \frac{\mathrm{d}}{\mathrm{d}\Omega^2}\sum_k \ln\left(1-\frac{\Omega^2+\gamma^2}{2\langle E_i\rangle\gamma}\langle m_k\rangle\right)$$

$$= -\frac{\mathrm{d}}{\mathrm{d}\Omega^2}\ln Q_1(\Omega). \tag{A6}$$

S_2 is best expressed in terms of the y_k:

$$S_2 = \sum_k \frac{2\gamma}{2\gamma+y_k^2+\gamma^2} = \sum_k \frac{\mathrm{d}}{\mathrm{d}\gamma}\ln\left(\frac{y_k^2+\gamma^2}{y_k^2}\right)$$

$$= -\frac{\mathrm{d}}{\mathrm{d}\gamma}\sum_k \ln\left(1-\frac{\langle m_k\rangle\gamma}{2\langle E_i\rangle}\right)$$

$$= -\frac{\mathrm{d}}{\mathrm{d}\gamma}\ln\prod_k\left(1-\frac{\langle m_k\rangle\gamma}{2\langle E_i\rangle}\right)$$

$$= +\frac{\mathrm{d}}{\mathrm{d}\gamma}[\ln\{e^{\gamma/2}Q_1(y=0)\}] = \tfrac{1}{2}. \tag{A7}$$

$Q_2(y)$ and $\hat{Q}_2(y)$ can thus be expressed in closed form. The final results, using (A3)–(A7), are

$$Q_2(y) = \exp\left(\frac{4\langle E_c\rangle \gamma s \mathscr{L}(y)}{Q_1{}^2(\Omega)}\right) \tag{A8}$$

$$\hat{Q}_2(y) = \exp\left(\frac{4\langle E_c\rangle \Omega^2 s M(y)}{\gamma Q_1{}^2(\Omega)}\right) \tag{A9}$$

where

$$\begin{aligned}\mathscr{L}(y) = {} & \frac{\gamma^2+2\gamma}{2(1-s\langle E_i\rangle)(\Omega^2+\gamma^2+2\gamma)} - \frac{s\langle E_i\rangle\gamma(\gamma^2+2s\langle E_i\rangle\gamma)\,\mathrm{d}\{\ln Q_1(y)\}/\mathrm{d}s}{y(1-s\langle E_i\rangle)(\Omega^2+\gamma^2+2s\langle E_i\rangle\gamma)^2} \\ & + \frac{\gamma\{(\Omega^2+\gamma^2)^2(\Omega^2-\gamma^2-2\gamma-2\gamma s\langle E_i\rangle)-4s\langle E_i\rangle\gamma^2(3\Omega^2+\gamma^2)\}\,\mathrm{d}\{\ln Q_1(\Omega)\}/\mathrm{d}\Omega}{2\Omega(\Omega^2+\gamma^2+2\gamma)^2(\Omega^2+\gamma^2+2s\langle E_i\rangle\gamma)^2} \\ & - \frac{\gamma(\Omega^2+\gamma^2)\,\mathrm{d}^2\{\ln Q_1(\Omega)\}/\mathrm{d}\Omega^2}{2(\Omega^2+\gamma^2+2\gamma)(\Omega^2+\gamma^2+2s\langle E_i\rangle\gamma)}\end{aligned} \tag{A10}$$

$$\begin{aligned}M(y) = {} & \frac{\gamma^2}{2(1-s\langle E_i\rangle)(\Omega^2+\gamma^2+2\gamma)^2} - \frac{s\langle E_i\rangle\gamma(\gamma^2+2s\langle E_i\rangle\gamma)\,\mathrm{d}\{\ln Q_1(y)\}/\mathrm{d}s}{y(1-s\langle E_i\rangle)(\Omega^2+\gamma^2+2s\langle E_i\rangle\gamma)^2} \\ & + \frac{\gamma\{(\Omega^2+\gamma^2)^2(\Omega^2-\gamma^2-2\gamma-2\gamma s\langle E_i\rangle)-4s\langle E_i\rangle\gamma^2(3\Omega^2+\gamma^2)\}\,\mathrm{d}\{\ln Q_1(\Omega)\}/\mathrm{d}\Omega}{2\Omega(\Omega^2+\gamma^2+2\gamma)^2(\Omega^2+\gamma^2+2s\langle E_i\rangle\gamma)^2} \\ & - \frac{\gamma(\Omega^2+\gamma^2)\,\mathrm{d}^2\{\ln Q_1(\Omega)\}\,\mathrm{d}\Omega^2}{2(\Omega^2+\gamma^2+2\gamma)(\Omega^2+\gamma^2+2s\langle E_i\rangle\gamma)}.\end{aligned} \tag{A11}$$

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